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Dirac particle in gravitational field of black hole with Newman-Unti-Tamburino parameter

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Abstract. We study the quantum-mechanical problem of a spin 1/2 particle in gravitational field of the black hole with Newman–Unti–Tamburino (NUT) parameter. Separation of the variables has been performed in the Dirac equation. The system of angular equations has been solved analytically in terms of hypergeometric functions. The quantization condition for the angular parameter is derived. In the massless case, the radial system is reduced to separate systems of two first-order equations which are solved in terms of the confluent Heun functions. Generalized tortoise-like coordinate is introduced, and expressions for the effective complex-valued potentials are found. We may expect that the present analysis can permit us to elucidate the physical interpretation for the Newman–Unti–Tamburino metric. The question arises about possible existence of the Hawking-like radiation in this model.

Keywords: black hole physics: Newman–Unti–Tamburino black hole; equations: individual (Dirac, Heun); spin 1/2 particle

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1 Introduction

Recently, the metrics with Newman–Unti–Tamburino (NUT) parameter have attracted the interest of scientific community. NUT parameter is usually interpreted as a gravitomagnetic charge (monopole) or as a linear source of a pure angular momentum (the twisting of the surrounding spacetime), see, e. g., Galtsov & Kobialko 2017; Chakraborty & Mukhopadhyay 2023. So, the black holes with NUT parameter are considered as physically meaningful systems with some special characteristics. Being axially-symmetric, the NUT black holes can exhibit new effects, such as asymmetry of black hole shadow or as the Lense–Thirring effect demonstrated by the rotating black holes (Ghasemi-Nodehi et al. 2021; Chakraborty & Mukhopadhyay 2023). The presence of the NUT-parameter in anti-de-Sitter metric leads to the appearance of a region with negative Gibbs free energy in thermodynamics of black holes that is associated to the phase transition (Awad & Elkhateeb 2023). The purpose of this work is to study the quantum-mechanical behavior of fermions in the background of the gravitational field of the NUT black hole.

2 Dirac equation in NUT space-time

To study a spin 1/2 particle in gravitational field of the black hole with NUT parameter, we search the solutions of the general covariant Dirac equation

$$\left[i\gamma^a\left(e^{\alpha}_{(a)}\frac{\partial}{\partial x^\alpha}+\frac{1}{2}\sigma^{mn}\gamma_{mna}\right)-M\right]\Psi=0. \tag{1}$$

Gravitational field of NUT black hole is took into account as the NUT spacetime background with metric

$$ds^2=\Phi\left(dt+4a\sin^2(\frac{\theta}{2})d\phi\right)^2-\frac{dr^2}{\Phi}-\left(a^2+r^2\right)\left(d\theta^2+\sin^2(\theta)d\phi^2\right),$$

where $\Phi=1-\frac{r_g r+2a^2}{r^2+a^2}$, a is a NUT parameter. We assume the Weyl basis and the bispinors are written in terms of two-component spinors as $\Psi=\begin{vmatrix}\xi\\ \chi\end{vmatrix}$, then the Dirac equation (1) takes the explicit form (let $\rho^2=r^2+a^2$, $\Delta=r^2-r_g r-a^2$, $\Phi=\frac{\Delta}{\rho^2}$)

$$\sigma_1\Big(\frac{1}{\rho}\chi_{,2}+\frac{1}{2\rho\tan\theta}\chi\Big)+\sigma_2\Big(\frac{1}{\rho\sin\theta}\chi_{,3}-\frac{2a}{\rho}\tan\frac{\theta}{2}\,\chi_{,0}\Big)$$

$$+\sigma_3\Big[\frac{\sqrt{\Delta}}{\rho}\chi_{,1}+\Big(\frac{\Delta'}{4\rho\sqrt{\Delta}}+\frac{\sqrt{\Delta}}{2\rho^3}\rho_-\Big)\chi\Big]+\frac{\rho}{\sqrt{\Delta}}\chi_{,0}+iM\xi=0,$$

$$\sigma_1\Big(\frac{1}{\rho}\xi_{,2}+\frac{1}{2\rho\tan\theta}\xi\Big)+\sigma_2\Big(\frac{1}{\rho\sin\theta}\xi_{,3}-\frac{2a}{\rho}\tan\frac{\theta}{2}\,\xi_{,0}\Big)$$

$$+\sigma_3\Big[\frac{\sqrt{\Delta}}{\rho}\xi_{,1}+\Big(\frac{\Delta'}{4\sqrt{\Delta}\rho}+\frac{\sqrt{\Delta}}{2\rho^3}\rho_+\Big)\xi\Big]-\frac{\rho}{\sqrt{\Delta}}\xi_{,0}-iM\chi=0;$$

where we apply the notations $\partial_\alpha =_{,\alpha}$, $\rho_+ = r+ia$, $\rho_- = r-ia$. To separate the variables we take into account the time independence and symmetry of the NUT spacetime and search spinors in the form $\xi=\Delta^{-1/4}\rho_+^{-1/2}e^{-i\epsilon t}e^{im\phi}X(r,\theta)$, $\chi=\Delta^{-1/4}\rho_-^{-1/2}e^{-i\epsilon t}e^{im\phi}Y(r,\theta)$, with the substitutions $X_1=R_1(r)T_1(\theta)$, $X_2=R_2(r)T_2(\theta)$, $Y_1=R_3(r)T_1(\theta)$, $Y_2=R_4(r)T_2(\theta)$. This procedure leads to angular equations

$$\frac{dT_1}{d\theta}+\Big(\frac{1}{2\tan\theta}-\frac{m}{\sin\theta}-2a\epsilon\tan\frac{\theta}{2}\Big)T_1-\Lambda T_2=0,$$

$$\frac{dT_2}{d\theta}+\Big(\frac{1}{2\tan\theta}+\frac{m}{\sin\theta}+2a\epsilon\tan\frac{\theta}{2}\Big)T_2-\Lambda T_1=0;$$

and the radial equations

$$\Big(\sqrt{\Delta}\frac{d}{dr}-\frac{ia\sqrt{\Delta}}{\rho^2}-\frac{i\epsilon\rho^2}{\sqrt{\Delta}}\Big)R_3+iM\rho_-R_1=\Lambda R_4,\ \Big(\sqrt{\Delta}\frac{d}{dr}-\frac{ia\sqrt{\Delta}}{\rho^2}+\frac{i\epsilon\rho^2}{\sqrt{\Delta}}\Big)R_4-iM\rho_-R_2=\Lambda R_3,$$
$$\Big(\sqrt{\Delta}\frac{d}{dr}+\frac{ia\sqrt{\Delta}}{\rho^2}+\frac{i\epsilon\rho^2}{\sqrt{\Delta}}\Big)R_1-iM\rho_+R_3=\Lambda R_2,\ \Big(\sqrt{\Delta}\frac{d}{dr}+\frac{ia\sqrt{\Delta}}{\rho^2}-\frac{i\epsilon\rho^2}{\sqrt{\Delta}}\Big)R_2+iM\rho_+R_4=\Lambda R_1.$$

The system of angular equations has been solved analytically in terms of hypergeometric functions G (let $z=\sin^2\frac{\theta}{2}$); there arises the quantization rule for the separation parameter Λ:

$$\Lambda^2=-(m+n_1+1/2)(m+n_1+1/2+4a\epsilon),\ m>0.$$

In the massless case, the radial system is reduced to separate systems of two first-order equations which are solved in terms of the confluent Heun functions $H(v)$; for instance the general solution for R_1 is

$$R_1=\frac{\epsilon^{1+i\epsilon r_g}e^{i\epsilon(r-r_1)}\,(r-r_1)^{\frac{1}{2}+i\epsilon r_1}\,(r-r_2)^{\frac{1}{2}+i\epsilon r_2}\sqrt{r+ia}}{\sqrt{r-ia}}H(v),\quad v=\frac{r-r_2}{r_1-r_2}. \qquad (2)$$

Analysis of the radial solution allows to conclude that the asymptotic behavior at infinity is not influenced by NUT parameter a, while near the exterior horizon the a-dependent phase shift arises.

Let find the effective potential (in massless case). In order to compare the present study with the results in Schwarzschild case, we introduce variables $f(x) = R_1 + R_2, g(x) = i(R_1 - R_2); x = \epsilon r$, and the generalized tortoise-like coordinate w

$$\frac{d^2w}{dx^2} + \Big[\frac{\Delta'}{\Delta} - \frac{2}{x+ia}\Big]\frac{dw}{dx} = 0, w = x + \frac{(a-ix_1)^2 \ln(x-x_1) - (a-ix_2)^2 \ln(x-x_2)}{x_2 - x_1},$$

which in the limiting case of $a = 0$ reduces to ordinary tortoise-like coordinate $w = x + \ln(x-1)$ (Chichurin et al. 2024). The second-order radial equations take the structure

$$\Big[\frac{d^2}{dw^2} + P(w)\Big]f = 0, \qquad \Big[\frac{d^2}{dw^2} + Q(w)\Big]g = 0.$$

Here $P(w)$ and $Q(w)$ are effective potentials. For instance, P is

$$P = \frac{(x-x_1)^2 (x-x_2)^2}{(a-ix)^4}\Big(\frac{(a^2+x^2)^2}{\Delta^2} + \frac{ia\Delta'}{\Delta(a^2+x^2)} + \frac{2\Lambda x}{\sqrt{\Delta}(a^2+x^2)} - \frac{a(a+4ix)}{(a^2+x^2)^2} - \frac{\Lambda\Delta'}{2\Delta^{3/2}} - \frac{\Lambda^2}{\Delta}\Big).$$

The dependence of the absolute value of the effective potential $P(x)$ is a decreasing monotonic curve similarly to the case of Schwarzschild metric, that evidence the absence of bounded states for such systems. However, $P(x)$ is a complex-valued function, which at the exterior horizon $x \to x_2$, $P = (x_2 - ia)^2/(x_2+ia)^2$; at the infinity $x \to \infty$, $P = 1$. It indicates the twisting character of the surrounding spacetime due to NUT-charge.

3 Summary

We may expect that the present analysis can permit us to elucidate the physical interpretation for the Newman–Unti–Tamburino metric. The question arises about possible existence of the Hawking-like radiation in this model.

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